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MAM1020S **Test 3: 50 min** **Spring 2017**

No credit will be given to unjustified answers (unless mentioned differently). Justify all your answers completely. (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could loose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...NO CALCULATOR OR MATERIAL ALLOWED

Problem 1: Advise for limit: learn how to justify a limit!

1. Use the polynomial division (factorization) to determine

$$lim_{x\to -2}\frac{x^3-x^2-10x-8}{5x^3+12x^2-2x-12}$$

2. Compute

$$lim_{t\to\infty}\frac{e^{-t}}{sin(t)+2}$$

(Hint: squeeze!)

Solution: We all know that for any real value x, $-1\leq sin(x)\leq 1$, thus $0<1\leq sin(x)+2\leq 3$. Finally,

$$\frac{1}{3}\leq\frac{1}{sin(x)+2}\leq 1$$

And since $e^{-x}>0$ we get

$$\frac{e^{-x}}{3}\leq\frac{e^{-x}}{sin(x)+2}\leq e^{-x}$$

We know that $lim_{x\to\infty}-x=-\infty$ and $lim_{X\to-\infty,\ X=-x}e^X=0$ thus using the composition limit rule we get

$$lim_{x\to\infty}e^{-x}=lim_{x\to\infty}\frac{e^{-x}}{3}=0.$$

Thus we can apply the squeeze theorem, that tells us that

$$lim_{x\to\infty}\frac{e^{-t}}{sin(t)+2}=lim_{x\to\infty}e^{-x}=lim_{x\to\infty}\frac{e^{-x}}{3}=0.$$

Be aware that the squeeze theorem is about squeezing a function in between two function having the SAME limit.

3. Use the limit definition of the derivative at a point to find:

$$lim_{\theta\to 1}\frac{tan^{-1}(\theta)-\pi/4}{tan(\pi/4\theta)-1}$$

Solution:
Note that

$$\frac{tan^{-1}(x)-\pi/4}{tan(\pi/4x)-1}=\frac{tan^{-1}(x)-\pi/4}{x-1}\frac{x-1}{tan(\pi/4x)-1}$$

we have $f:x\mapsto tan^{-1}(x)$ and $g:x\mapsto tan(\pi/4x)$ are differentiable at 1, and $f'(x)=\frac{1}{x^2+1}$ and $g'(x)=\pi/4sec^2(\pi/4x)$ thus by the definition of the derivative, we have

$$lim_{x\to 1}\frac{tan^{-1}(x)-\pi/4}{x-1}=f'(1)=\frac{1}{1^2+1}=1/2$$

and

$$lim_{x\to 1}\frac{tan(\pi/4x)-1}{x-1} = g'(1) = \pi/4sec^2(\pi/4) = \pi/8$$

Thus,

$$lim_{x\to 1}\frac{tan^{-1}(x)-\pi/4}{tan(\pi/4x)-1} = lim_{x\to 1}\frac{tan^{-1}(x)-\pi/4}{x-1}\cdot\frac{1}{lim_{x\to 1}\frac{tan(\pi/4x)-1}{x-1}} = 1/2\cdot 8/\pi = 4/\pi$$

Remember that if you type $\pi/4\theta$ is a calculator, it is doing by the rules of multiplication and division $\frac{\pi}{4}\theta$ if you want it to do $\frac{\pi}{4\theta}$ you will need to put parenthesis around (4θ). When writing mathematics we are always applying those rules.

Problem 2

1. Use logarithm differentiation to compute the derivative of

$$f(t) = ln\left(\frac{t^4}{(2t-1)^3(t^2+1)}\right)$$

Solution:

Using the logarithm rules, (step two of logarithm differentiation we have:

$$f(t) = ln\left(\frac{t^4}{(2t-1)^3(t^2+1)}\right) = 4ln(t) - 3ln(2t-1) - ln(t^2+1)$$

Then, using three time the chain rule for logarithm we get.

$$f'(t) = \frac{4}{t} - \frac{6}{2t-1} - \frac{2t}{t^2+1}$$

Advise: Make your life easier before you differentiate.

2. Differentiate

$$g(\theta) = sec(3\theta)csc(5\theta) + e^5$$

Solution:

Using the product rule and twice the chain rule, we get

$$f'(x) = 3sec(3x)tan(3x)csc(5x) - 5sec(3x)csc(5x)cot(5x)$$

Be aware that e^5 is a constant as anything that does not depends on x (like $\sqrt{5}+2$ or $\sqrt[4]{5}+ln(6)+10$ ...) and thus its derivative is 0 !!!! Please be careful with this next time.

3. Differentiate

$$h(x) = \frac{(3x^3-1)^4}{(x^4+x)^{2x-1}}$$

Solution: The quicker way here would be to do a logarithm differentiation. That is apply ln in both side of the equality first:

$$ln(h(x)) = ln\left(\frac{(3x^3-1)^4}{(x^4+x)^{2x-1}}\right)$$

Then apply logarithm rules **UNTIL you get the simplest form for you to differentiate (remember the logarithm rules are $ln(a+b) = ln(a)+ln(b)$, $ln(\frac{a}{b}) = ln(a)-ln(b)$, $ln(a^b) = bln(a)$.)** Doing this we get

$$ln(h(x)) = ln(3x^3-1)^4 - ln(x^4+x)^{2x-1}$$

DO NOT STOP here for instance otherwise why did we do logarithm differentiation? Use all the logarithm rule... as show in the next line.

$$ln(h(x)) = ln(3x^3 - 1)^4 - ln(x^4 + x)^{2x-1} = ln(h(x)) = 4ln(3x^3 - 1) - (2x - 1)ln(x^4 + x)$$

Since we cannot use any logarithm rule anymore we now differentiate. But please remember you have two side in your equality.... So you need to differentiate BOTH.

Doing that, we obtain using the chain rule and the product rule:

$$\frac{h'(x)}{h(x)} = 4\frac{9x^2}{3x^3 - 1} - (2ln(x^4 + x) + (2x - 1)\frac{4x^3 + 1}{x^4 + x})$$

DO NOT FORGET, you want $h'(x)$ not $\frac{h'(x)}{h(x)}$ so do not stop here!!!!

We have

$$\begin{array}{rcl} h'(x) & = & h(x) \cdot (4\frac{9x^2}{3x^3-1} - 2ln(x^4 + x) - (2x - 1)\frac{4x^3+1}{x^4+x}) \\ & = & \frac{(3x^3-1)^4}{(x^4+x)^{2x-1}} \cdot (\frac{36x^2}{3x^3-1} - 2ln(x^4 + x) - (2x - 1)\frac{4x^3+1}{x^4+x}) \end{array}$$

Be aware that the power rule only apply if the power DOES NOT DEPEND ON x thus here it does not apply for $(x^4 + x)^{2x-1}$!!!!!

Problem 3 The area of a spherical bubble is increasing at 2 cm^2/s. When the radius of the bubble is 6 cm, at what rate is the volume of the bubble increasing?

Solution:
YOU should consider to do a draft for this kind of question where you have to think!!!
Let's first fix the notation in an intuitive way (**VERY IMPORTANT STEP BY THE WAY**).
Let A be the area of the bubble, r be the radius and V the volume.
We know that

$$\frac{dA}{dt} = 2\ cm^2/2$$

We want to know

$$\frac{dV}{dt}$$

when $r = 6cm$.
Now, we all know (maybe) that

$$A = 4\pi r^2$$

and

$$V = \frac{4}{3}\pi r^3$$

Thus using the chain rule we get that

$$\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$$

Also we have that

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = \frac{1}{2}8\pi r\frac{dr}{dt}r = \frac{1}{2}r\frac{dA}{dt} = \frac{1}{2} \cdot 6 \cdot 2 = 6cm^3/sec$$

Super easy no? when you take it easy and you do a draft and try to think of the best way to get an answer quickly without too much computation.
Problem 4: Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x + 1)e^{2x}$.

1. Compute the domain of definition of f

   f is a the product of a polynomial with an exponential composed with a polynomial. All of those function are know to be defined for any real number thus the domain of definition of f is also the set of all real numbers.

2. Compute the x-intercept and y-intercept for f.

   The x-intercept is the points of the graph of your curve crossing with the x-axis that is whose ordinate is 0. So we can find their abscissa by solving the equation:

   $$f(x) = 0$$

   That is,

   $$(x+1)e^{2x} = 0$$

   Since $e^{2x} > 0$ then

   $$x + 1 = 0 \Leftrightarrow x = -1$$

   Thus the only x-intercept is $(-1, 0)$.

   The y-intecept is the point of the graph of your curve crossing with the y-axis that is whose abscissa is 0. We can thus find its ordinate by computing

   $$f(0) = (0+1)e^{2\cdot 0} = 1$$

   Thus the y-intercept is $(0, 1)$.

   REMEMBER that an exponential is always strictly positive thus $e^{WHATEVER} = 0$ has NEVER A SOLUTION. Also $e^0 = 1$!!!! and $ln(0)$ DOES NOT EXIST!!!!

3. Compute $lim_{x\to\infty} f(x)$

   Clearly $lim_{x\to\infty}(x+1) = \infty$ and $lim_{x\to\infty} 2x = \infty$, also $lim_{x\to\infty} e^X = \infty$, thus composing limit we obtain

   $$lim_{x\to\infty} e^{2x} = \infty$$

   Finally we use the product rules for limits and obtain

   $$lim_{x\to\infty} f(x) = \infty$$

   I DO NOT WANT TO SEE YOU multiplying and dividing with infinity. Please do not write $\infty \cdot \infty$ or $\infty \cdot 0$ or .... as some of them even do not make sense and we have not defined those. Rathe do a sentence in english. Also tell me which limit rule you are using that will be part of your justification. And I can see why why answer is what it is and it is also beneficial for you as you are making sure that your answer make sense.)

4. Compute $lim_{x\to-\infty} f(x)$

   Solution: Here if you try doing just the limit right away you find an indeterminate. If you do not see anything else to do you can always think of trying out the hospital rule. Do not try to write an answer just to write an answer most of the time you will get 0 try at least to explain it. The hospital rule requires quite a bit of condition and need to be stated on a quotient. Thus you might need to write first the function as a quotient and then go from there.

   We write

   $$f(x) = \frac{(x+1)}{e^{-2x}}$$

   We have that the function u defined by $u(x) = x + 1$ and the function v defined by $v(x) = e^{-2x}$ are differentiable over $\mathbb{R}$ and $lim_{x\to-\infty}(x+1) = -\infty$ and $lim_{x\to-\infty} - 2x = \infty$, also $lim_{x\to\infty} e^X = \infty$, thus composing limit we obtain

   $$lim_{x\to-\infty} e^{-2x} = \infty$$

   We thus obtain ∞ over ∞ which is an undeterminate form for the quotient. We have therefore all the condition required for the Hospital rule to apply and since $u'(x) = 1$ and $v'(x) = -2e^{-2x}$, we get

   $$lim_{x\to-\infty} f(x) = lim_{x\to-\infty} \frac{1}{-2e^{-2x}} = -e^{2x}/2$$

Note that $lim_{x\to-\infty}2x = -\infty$, also $lim_{x\to-\infty}e^X = 0$, thus composing limit we obtain

$$lim_{x\to-\infty}e^{2x} = 0$$

Hence,

$$lim_{x\to-\infty}f(x) = 0$$

Do not forget how to justify composite function as done above!!!

5. Deduce asymptotes for the graph of f from the previous questions.

   From $lim_{x\to-\infty}f(x) = 0$, we get that $y = 0$, the x-axis is an asymptote for the graph of f.

6. Compute $f'(x)$.

   Using the product rule, we get
   $$f'(x) = 1 \cdot e^{2x} + 2(x+1)e^{2x}$$

   Why would you stop here? Please think that when you compute is derivative it is to decide when f is increasing and decreasing. Thus you will need to find the sign of the derivative. Thus you need to prepare for that so to prepare for that you will need to write $f'(x)$ as a product, quotient of functions and know that you can determine the sign of each one of them. Try to anticipate !!!! Read the next questions!! It could help you to write the answer in its best form.

   Here it is not hard to just do

   $$f'(x) = 1 \cdot e^{2x} + 2(x+1)e^{2x} = (2x+3)e^{2x}$$

7. Deduce over which interval f is increasing/decreasing.

   Since $e^x > 0$, the sing of $f'(x)$ is the same as the one of $(2x+3)$. And $2x+3 > 0$ when $x > -3/2$, thus $f'(x) > 0$ when $x > -3/2$ and f is increasing over $(-3/2, \infty)$. Also, $2x+3 < 0$ when $x < -3/2$, thus $f'(x) < 0$ when $x < -3/2$ and f is decreasing over $(-\infty, -3/2)$.

8. Compute the critical point.

   A critical number is a number x such that $f'(x)$ does not exist or $f'(x) = 0$.

   Since f is differentiable over $\mathbb{R}$ as product, composite of function differentiable over $\mathbb{R}$. Thus $f'(x)$ always exists. We are thus left with solving $f'(x) = 0$ That is,

   $$(2x+3)e^{2x} = 0$$

   Since $e^{2x} > 0$, then this is equivalent to
   $$2x + 3 = 0$$

   that is $x = -3/2$.

   And we obtain an unique critical point $(-3/2, f(-3/2))$ where

   $$f(-3/2) = (-3/2+1)e^{2(-3/2)} = -1/2e^{-3}.$$

9. Determine the local minimum/maximum.

   Local maximum/ minimum are determined by the critical point is the variation of f changes at this point. Since, $f'(x) > 0$ when $x > -3/2$ and $f'(x) < 0$ when $x < -3/2$, thus the critical point found before $(-3/2, f(-3/2))$ is a local minimum.

   You need to show that the derivative changes sign at your critical point and how it changes sign to deduce it is a local minimum or maximum.

10. Compute $f''(x)$

Using the product rule and chain rule on f' we have:

$$f''(x) = 2e^{2x} + 2(2x+3)e^{2x} = 4(x+2)e^{2x}$$

Same as before look at the next question you need to find the concavity of your function thus you need to find the sign of f''. Thus factorize as much as you can.

11. Deduce where f is concave upward/downward

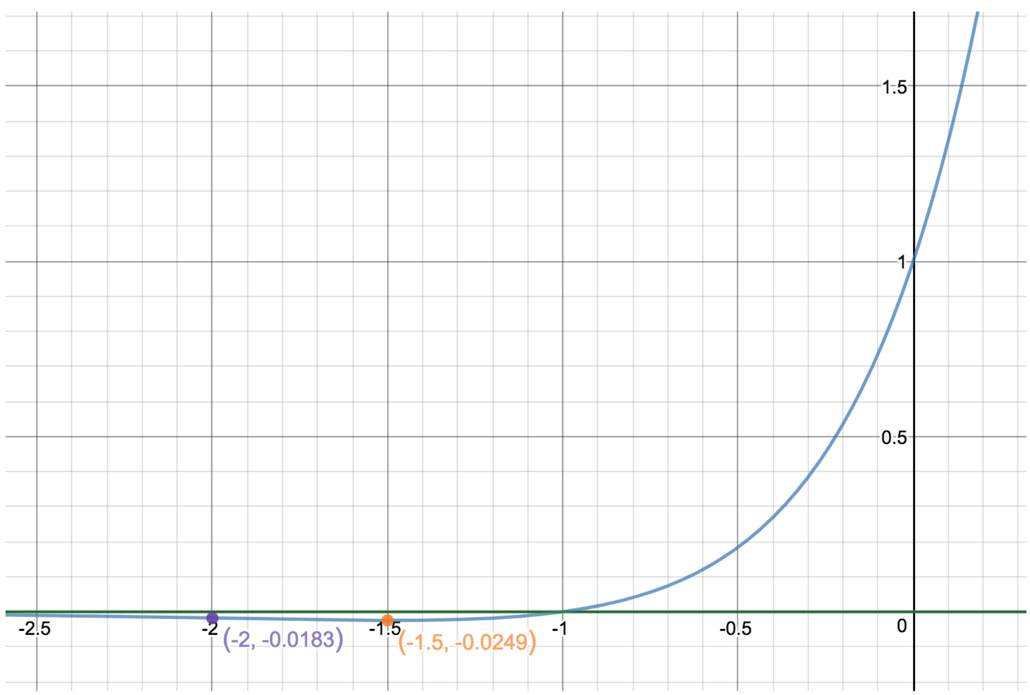
Since $4e^{2x} > 0$, the sign of $f''(x)$ is the same as the sign of $x+2$ and $x+2 > 0$ if and only if $x > -2$ thus $f''(x) > 0$ when $x > -2$ hence, f is concave upward over $(-2, \infty)$ and $x + 2 < 0$ if and only if $x < -2$ thus $f''(x) < 0$ when $x < -2$ hence f is concave downward over $(-\infty, -2)$.

12. Deduce the inflection point.

An inflection number is a number x where the function changes concavity. We see that $x = -2$ is the only number where f changes concavity since f is concave upward over $(-2, \infty)$ and concave downward over $(-\infty, -2)$. Thus $(-2, f(-2))$ is the only inflection point where

$$f(-2) = (-2+1)e^{2(-2)} = -e^{-4}$$

13. Sketch a quick graph of f with all the information above.



Advise for final:

- **Read the questions following a question on a same problem, it could help you to go further.**
- **Write the definition of the element you are trying to identify, at least I know you know them. You need to show me that you know staff, I cannot read your mind, the only way that I can know that you know is by you showing it to me. Example: local minimal point are determined by critical point where the derivative pass from negative to positive. If you do not say any part of this you are missing something. Other example: inflection point**

a point where your function changes concavity, if you do not tell me that your function changes concavity and that is why you wrote what you wrote. How do I differentiate you from someone who do not know the definition and wrote also what you wrote?

- **There is no universal rule, so take a breath before any question observe your question and think what can I do to make it simpler, can I do something? Example for derivative: use logarithm rules, differential rule, change the form of your initial function. You do not have to go straight to the answer sometimes there is a preliminary work which could make your life easier.**
- **Use your draft, an exam is not a draft.**
- **Write english understandable sentences!!**